Dynamic programming: a fully worked out example

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Abstract

This document was written upon request of some of the students in the class who wanted to see a dynamic programming example fully worked out. Your job as a CE191 student is to check that all the indices and the values in this document are correct!

Consider N aircraft labelled by the index $i \in \{1, \cdots, N\}$. The arrival time τ_i of aircraft i belongs to the set of n_i feasible arrival times, called $t_{i,j}$, indexed by $j \in \{1, \cdots, n_i\}$. We will assume that the following condition holds: for all i and i' such that $i > i'$, $t_{i,j} > t_{i',j'}$ for all j and j'. This situation is represented in Figure 1 This means that no matter what choice the algorithm makes, aircraft always arrive in increasing order of i. The algorithm presented below can easily be generalized when this condition is not met, but the order of arrival of aircraft is still in increasing order of i.

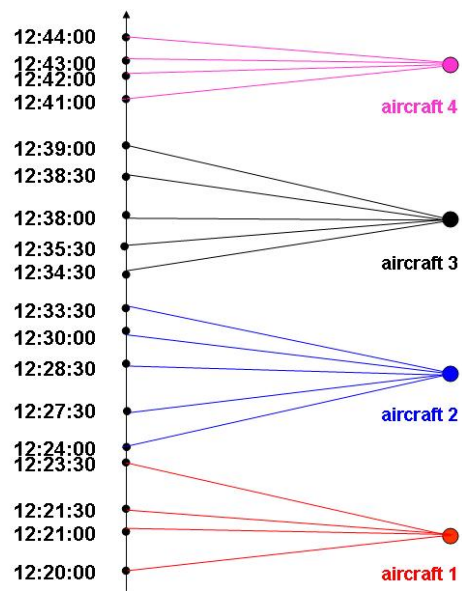

Figure 1: Example of set of $t_{i,j}$ such that for all i and i' such that $i > i'$, $t_{i,j} > t_{i',j'}$ for all j and j'. Note that the scale on the left axis is dummy.

The optimization program that we are trying to solve with dynamic programming is as follows:

$$\begin{array}{lll} \textbf{max:} & \Delta & \\ \textbf{s.t.:} & \tau_i \in \{t_{i,j}\}_{j \in \{1,\cdots,n_i\}} & \forall i \in \{1, \cdots, N\} \\ & \tau_i - \tau_{i-1} \geq \Delta & \forall i \in \{2, \cdots, N\} \end{array} \tag{1}$$

In the previous program, Δ represents the maximum minimum spacing between aircraft upon landing. For example, if $\Delta = 3$ minutes, this means that among the N aircraft that will land, there are two of them that are separated by 3 minutes, and that this is the best spacing that Air Traffic Control can achieve. For example, if $N = 4$ aircraft, and the landing times are $\tau_1 = 12:30:00$, $\tau_2 = 12:34:00$, $\tau_3 = 12:37:00$, and

$\tau_3 = 12:42:00$, $\Delta = 3$ minutes because $\tau_3 - \tau_2 = 3$ minutes. Note that because the aircraft arrive in increasing order, i.e. 1 before 2, 2 before 3, 3 before 4, etc., the condition

$$|\tau_i - \tau_{i-1}| \geq \Delta$$

boils down to

$$\tau_i - \tau_{i-1} \geq \Delta$$

This problem can be solved exactly with dynamic programming:

Proposition 1 *The solution Δ to the optimization program (1) is given by $\Delta = \delta(N, n_N)$, where, $\delta(N, n_N)$ is defined by the following recurrence:*

$$\begin{aligned} \delta(2,j) &= t_{2,j} - t_{1,1} \\ \delta(i,j) &= \max_{j'=1}^{n_{i-1}} \{\min\{t_{i,j} - t_{i-1,j'}\ ,\ \delta(i-1,j')\}\} \end{aligned} \tag{2}$$

We now illustrate this dynamic programming algorithm with a fully worked out example. Without loss of generality, let us assume that the arrival times are given in minutes, starting from an arbitrary time:

- **Aircraft 1:** $t_{1,1} = 0$, $t_{1,1} = 3$, $t_{1,1} = 5$, thus $n_1 = 3$, since there are 3 possible arrival times for aircraft 1.
- **Aircraft 2:** $t_{2,1} = 11$, $t_{2,2} = 12$, $t_{2,3} = 13$, $t_{2,4} = 14$, $t_{2,5} = 15$, thus $n_2 = 5$, since there are 4 possible arrival times for aircraft 2.
- **Aircraft 3:** $t_{3,1} = 16$, $t_{3,2} = 19$, thus $n_3 = 2$, since there are 2 possible arrival times for aircraft 3.
- **Aircraft 4:** $t_{4,1} = 25$, $t_{4,2} = 26$, $t_{4,3} = 30$, thus $n_4 = 3$, since there are 3 possible arrival times for aircraft

Step 1 (initialization step). The initialization step is trivial. First, we scheduled aircraft 1 at $t_{1,1} = 0$, since obviously, we should have it arrive as soon as possible. Then, we compute $\delta(2,j)$, i.e. the maximum minimum spacing between aircraft 1 and 2, if aircraft 2 is scheduled at time $t_{2,j}$, and aircraft 1 is scheduled at time $t_{1,1} = 0$. This is in fact the spacing between aircraft 1 and 2, if aircraft 2 is scheduled at time $t_{2,j}$ (i.e. no need for the maximum minimum).

Trivially, we have
- $\delta(2,1) = t_{2,1} - t_{1,1} = 11$
- $\delta(2,2) = t_{2,2} - t_{1,1} = 12$
- $\delta(2,3) = t_{2,3} - t_{1,1} = 13$
- $\delta(2,4) = t_{2,4} - t_{1,1} = 14$
- $\delta(2,5) = t_{2,5} - t_{1,1} = 15$

Note that we have performed $n_2 = 5$ computations for this step.

Step 2. Now, we have to add aircraft 3. Aircraft 3 has two options: $t_{3,1} = 16$, and $t_{3,2} = 19$. This means that we need to compute two values of δ:
- $\delta(3,1)$, i.e. the maximum minimum spacing between any two of the three aircraft if aircraft 3 arrives at $t_{3,1}$ (its first possible arrival time)
- $\delta(3,2)$, i.e. the maximum minimum spacing between any two of the three aircraft if aircraft 3 arrives at $t_{3,2}$ (its second possible arrival time).

Let us compute $\delta(3,1)$. This is the case in which aircraft 3 arrives at $t_{3,1} = 16$. In this scenario, we have the following schedules, corresponding to the $n_2 = 5$ options of aircraft 2:
- $(t_{1,1}, t_{2,1}, t_{3,1}) = (0, 11, 16)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,1) = t_{2,1} - t_{1,1} = 11$, $t_{3,1} - t_{2,1} = 16 - 11 = 5$.
- $(t_{1,1}, t_{2,2}, t_{3,1}) = (0, 12, 16)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,2) = t_{2,2} - t_{1,1} = 12$, $t_{3,1} - t_{2,2} = 16 - 12 = 4$.
- $(t_{1,1}, t_{2,3}, t_{3,1}) = (0, 13, 16)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,3) = t_{2,3} - t_{1,1} = 13$, $t_{3,1} - t_{2,3} = 16 - 13 = 3$.
- $(t_{1,1}, t_{2,4}, t_{3,1}) = (0, 14, 16)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,4) = t_{2,4} - t_{1,1} = 14$, $t_{3,1} - t_{2,4} = 16 - 14 = 2$.

- $(t_{1,1}, t_{2,5}, t_{3,1}) = (0, 15, 16)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,5) = t_{2,5} - t_{1,1} = 15$, $t_{3,1} - t_{2,5} = 16 - 15 = 1$.

Clearly, the best option so far is is the first, since the smallest (minimum) separation occurs between aircraft 2 and 3, and the best option (maximum) is 5. Therefore, $\delta(3,1) = 5$, for aircraft 1 scheduled at $t_{1,1} = 0$, aircraft 2 scheduled at $t_{2,1} = 11$. If you want to apply the formula (2), you will obtain the same result:

$$\delta(3,1) = \max_{j'=1}^{n_2} \{\min\{t_{3,1} - t_{2,j'}\ ,\ \delta(2,j')\}\} = \max_{j'=1}^{5} \{\min\{16 - t_{2,j'}\ ,\ \delta(2,j')\}\}$$

This can be rewritten as:

$$\delta(3,1) = \max\{\min\{11,5\}, \min\{12,4\}, \min\{13,3\}, \min\{14,2\}, \min\{15,1\}\} = \max\{5,4,3,2,1\} = 5$$

Let us now compute $\delta(3,2)$. This is the case in which aircraft 3 arrives at $t_{3,2} = 19$. In this scenario, we have the following schedules, corresponding again to the $n_2 = 5$ options of aircraft 2:

- $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,1) = t_{2,1} - t_{1,1} = 11$, $t_{3,2} - t_{2,1} = 19 - 11 = 8$.
- $(t_{1,1}, t_{2,2}, t_{3,2}) = (0, 12, 19)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,2) = t_{2,2} - t_{1,1} = 12$, $t_{3,2} - t_{2,2} = 19 - 12 = 7$.
- $(t_{1,1}, t_{2,3}, t_{3,2}) = (0, 13, 19)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,3) = t_{2,3} - t_{1,1} = 13$, $t_{3,2} - t_{2,3} = 19 - 13 = 6$.
- $(t_{1,1}, t_{2,4}, t_{3,2}) = (0, 14, 19)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,4) = t_{2,4} - t_{1,1} = 14$, $t_{3,2} - t_{2,4} = 19 - 14 = 5$.
- $(t_{1,1}, t_{2,5}, t_{3,2}) = (0, 15, 19)$, then the separations between aircraft 1 and 2, 2 and 3 are respectively $\delta(2,5) = t_{2,5} - t_{1,1} = 15$, $t_{3,2} - t_{2,5} = 19 - 15 = 4$.

Again, the dynamic programming formula gives us:

$$\delta(3,2) = \max\{\min\{11,8\}, \min\{12,7\}, \min\{13,6\}, \min\{14,5\}, \min\{15,4\}\} = \max\{8,7,6,5,4\} = 8$$

In total, we have performed $2 \times 5 = n_3 \times n_2$ computations for this step.

Step 3. Now, we have computed the following deltas: $\delta(2,1) = 11$, $\delta(2,2) = 12$, $\delta(2,3) = 13$, $\delta(2,4) = 14$, $\delta(2,5)$=15, $\delta(3,1) = 5$, $\delta(3,2) = 8$.

$\delta(3,1) = 5$ corresponds to the sequence $(t_{1,1}, t_{2,1}, t_{3,1}) = (0, 11, 16)$.
$\delta(3,2) = 8$ corresponds to the sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$.

We want to schedule the last aircraft (aircraft 4). There are three options for this aircraft: $t_{4,1} = 25$, $t_{4,2} = 26$ and $t_{4,1} = 30$. Therefore, we have to compute $\delta(4,1)$, $\delta(4,2)$, $\delta(4,3)$, corresponding to these three options.

Let us now compute $\delta(4,1)$. This is the case in which aircraft 3 arrives at $t_{4,1} = 25$. In this scenario, we have the following schedules, corresponding again to the $n_3 = 2$ options of aircraft 3:

- Option 1: aircraft 3 arrives at $t_{3,1} = 16$. This corresponds to $\delta(3,1) = 5$ with arrival sequence $(t_{1,1}, t_{2,1}, t_{3,1}) = (0, 11, 16)$. The new resulting delta is thus the min of $\delta(3,1) = 5$ and $t_{4,1} - t_{3,1} = 25 - 16 = 9$, therefore 5.
- Option 2: aircraft 3 arrives at $t_{3,2} = 19$. This corresponds to $\delta(3,2) = 8$ with arrival sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$. The new resulting delta is thus the min of $\delta(3,2) = 8$ and $t_{4,1} - t_{3,2} = 25 - 19 = 6$, therefore 6.

In summary, $\delta(4,1)$ is given by

$$\delta(4,1) = \max\{\min\{5,9\}, \min\{8,6\}\} = \max\{5,6\} = 6$$

Let us now compute $\delta(4,2)$. This is the case in which aircraft 3 arrives at $t_{4,2} = 26$. In this scenario, we have the following schedules, corresponding again to the $n_3 = 2$ options of aircraft 3:

- Option 1: aircraft 3 arrives at $t_{3,1} = 16$. This corresponds to $\delta(3,1) = 5$ with arrival sequence $(t_{1,1}, t_{2,1}, t_{3,1}) = (0, 11, 16)$. The new resulting delta is thus the min of $\delta(3,1) = 5$ and $t_{4,2} - t_{3,1} = 26 - 16 = 10$, therefore 5.
- Option 2: aircraft 3 arrives at $t_{3,2} = 19$. This corresponds to $\delta(3,2) = 8$ with arrival sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$. The new resulting delta is thus the min of $\delta(3,2) = 8$ and $t_{4,2} - t_{3,2} = 26 - 19 = 7$, therefore 7.

In summary, $\delta(4,2)$ is given by

$$\delta(4,2) = \max\{\min\{5,10\}, \min\{8,7\}\} = \max\{5,7\} = 7$$

Let us now compute $\delta(4,3)$. This is the case in which aircraft 3 arrives at $t_{4,3} = 30$. In this scenario, we have the following schedules, corresponding again to the $n_3 = 2$ options of aircraft 3:

- Option 1: aircraft 3 arrives at $t_{3,1} = 16$. This corresponds to $\delta(3,1) = 5$ with arrival sequence $(t_{1,1}, t_{2,1}, t_{3,1}) = (0, 11, 16)$. The new resulting delta is thus the min of $\delta(3,1) = 5$ and $t_{4,3} - t_{3,1} = 30 - 16 = 14$, therefore 5.
- Option 2: aircraft 3 arrives at $t_{3,2} = 19$. This corresponds to $\delta(3,2) = 8$ with arrival sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$. The new resulting delta is thus the min of $\delta(3,2) = 8$ and $t_{4,3} - t_{3,2} = 30 - 19 = 11$, therefore 8.

In summary, $\delta(4,3)$ is given by

$$\delta(4,3) = \max\{\min\{5,14\}, \min\{8,11\}\} = \max\{5,8\} = 8$$

The final result for aircraft 4 is thus:

- $\delta(4,1) = 6$ with sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$ and $t_{4,1} = 25$, therefore $(t_{1,1}, t_{2,1}, t_{3,2}, t_{4,1}) = (0, 11, 19, 25)$. The corresponding shortest Δ is between aircraft 3 and 4.
- $\delta(4,2) = 7$ with sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$ and $t_{4,2} = 26$, therefore $(t_{1,1}, t_{2,1}, t_{3,2}, t_{4,2}) = (0, 11, 19, 26)$. The corresponding shortest Δ is between aircraft 3 and 4.
- $\delta(4,3) = 8$ with sequence $(t_{1,1}, t_{2,1}, t_{3,2}) = (0, 11, 19)$ and $t_{4,3} = 30$, therefore $(t_{1,1}, t_{2,1}, t_{3,2}, t_{4,3}) = (0, 11, 19, 30)$. The corresponding shortest Δ is between aircraft 2 and 3.

Note that the number of computation performed is $2 \times 3 = n_3 \times n_4$.

We now summarize all the deltas that we have computed so far: the $\delta(i,j)$ are summarized in the table below:

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	N.A.	N.A.	N.A.	N.A.	N.A.
$i=2$	11	12	13	14	15
$i=3$	5	8	N.A.	N.A.	N.A.
$i=4$	6	7	8	N.A.	N.A.

Now we can finally solve our problem: we wanted to schedule 4 aircraft, and we wanted for these 4 aircraft the best possible Δ. This is 8, obtained for $\delta(4,3)$, which means aircraft 4 is scheduled at $t_{4,3}$. This corresponds to $(t_{1,1}, t_{2,1}, t_{3,2}, t_{4,3}) = (0, 11, 19, 30)$.

Remark 1. Note that the total number of computations performed is $n_2 + n_3 \times n_2 + n_4 \times n_3 = 5 + 2 \times 5 + 3 \times 2 = 21$. If we had blindly investigated all possibilities, we would have performed $n_1 \times n_2 \times n_3 \times n_4 = 3 \times 5 \times 2 \times 3 = 90$ computations. The power of this method appears clearly if you have large numbers of aircraft, for example if you had 10 aircraft with two possibilities, the cost of this method would be

$$n_2 + \sum_{i=3}^{10} n_i \times n_{i-1} = 2 + 8 \times 4 = 34$$

The cost of investigating all possibilities would be $2^{10} = 1024$, which is now significantly higher!!!

Remark 2. The total number of $\delta(i,j)$ that you need to compute is $n_1 + n_2 + n_3 + n_4$: the other boxes in the table above are empty, since they do not have any meaning. For example $\delta(3,5)$ does not make any sense, since there is no $t_{3,5}$ (aircraft 3 has only two options). So the total of number of $\delta(i,j)$ that we need to compute with this method is:

$$\sum_{i=2}^{N} n_i = 5 + 2 + 3 = 10$$

where N is the total number of aircraft

Remark 3. We do not always need to schedule the last aircraft last. For example, assume that now aircraft 4 has 5 possible arrival time, with $t_{4,4} = 31$ and $t_{4,5} = 32$. Then (exercise), check that the table becomes

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	N.A.	N.A.	N.A.	N.A.	N.A.
$i = 2$	11	12	13	14	15
$i = 3$	5	8	N.A.	N.A.	N.A.
$i = 4$	6	7	8	8	8

In this case, you can still see that it does not help to use $t_{4,4} = 31$ or $t_{4,5} = 32$ because the smallest delta happens between aircraft 2 and aircraft 3, i.e. $t_{3,2} - t_{2,1} = 19 - 11 = 8$.